Indexed Programming

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1. Collections

```
public interface List {
   void add (int index, Object element);
   boolean contains (Object o);
   Object get (int index);
   int size ();
}
```

Loss of precision:

```
List l = new ArrayList ();    //  ...which implements List
String s = "abc";
l.add (0, s);                 //  upcasting when inserting
s = (String) l.get (0);       //  downcasting when retrieving
```

2. Generics

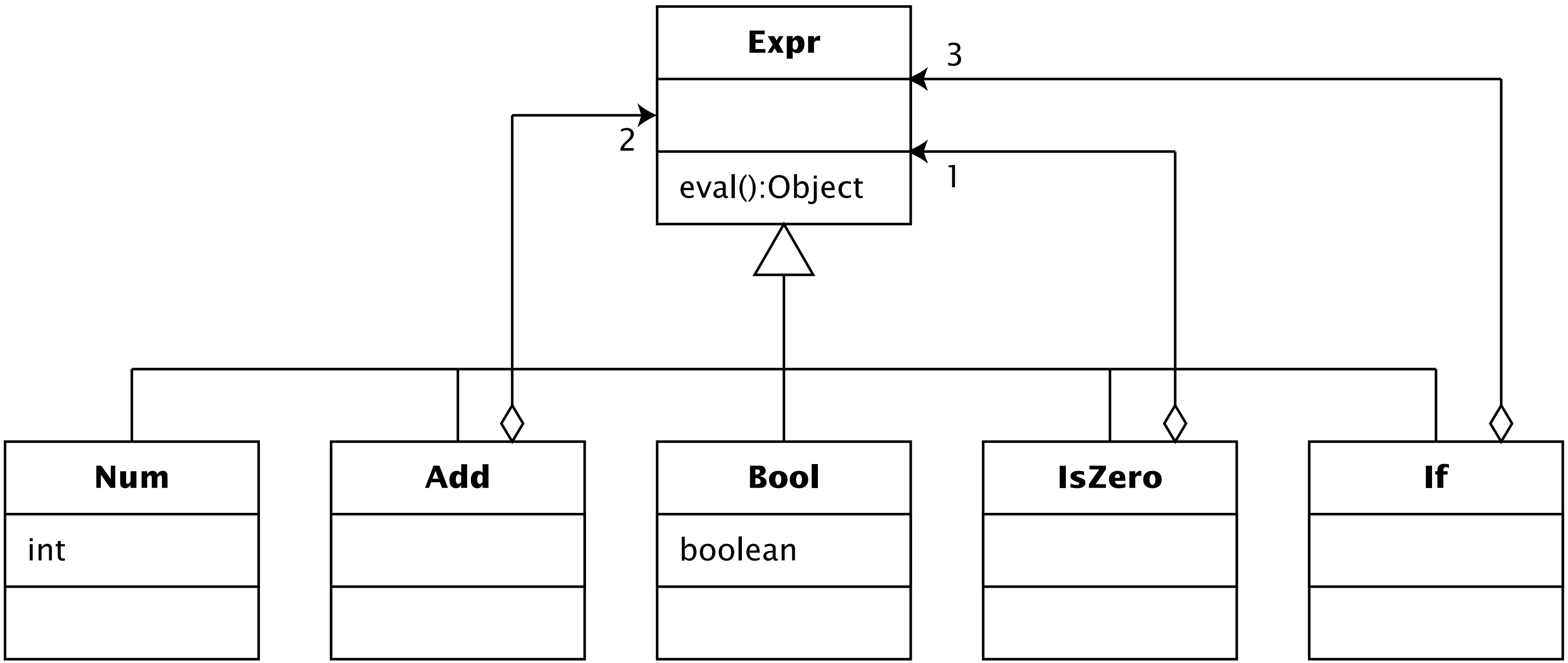
$$
\begin{array}{l}
\textbf{public interface } \mathit{List}\langle E\rangle\ \{\\
\quad \textbf{void } \mathit{add}\ (\textbf{int } \mathit{index}, E\ \mathit{element});\\
\quad \textbf{boolean } \mathit{contains}\ (\mathit{Object}\ o);\\
\quad E\ \mathit{get}\ (\textbf{int } \mathit{index});\\
\quad \textbf{int } \mathit{size}\ ();\\
\}
\end{array}
$$

More precise code:

$$
\begin{array}{ll}
\mathit{List}\langle \mathit{String}\rangle\ l = \mathit{new}\ \mathit{ArrayList}\langle \mathit{String}\rangle(); & \\
\mathit{String}\ s = \texttt{"abc"}; & \\
l.\mathit{add}\ (0, s); & //\quad \text{no upcasting or}\ldots\\
s = l.\mathit{get}\ (0); & //\quad \ldots\text{downcasting needed}
\end{array}
$$

This is called *parametric polymorphism*.

3. Algebraic datatypes



3.1. Expression datatype in Java

```
public abstract class Expr {
  public abstract Object eval ();
}

public class Num extends Expr {
  private Integer n;
  public Num (Integer n) { this.n = n; }
  public Object eval () { return n; }
}

public class Add extends Expr {
  private Expr x, y;
  public Add (Expr x, Expr y) {
    this.x = x; this.y = y;
  }
  public Object eval () {
    return (Integer) (x.eval ()) + (Integer) (y.eval ());
  }
}

public class Bool extends Expr {
  private Boolean b;
  public Bool (Boolean b) { this.b = b; }
  public Object eval () { return b; }
}
```

```
public class IsZero extends Expr {
  private Expr e;
  public IsZero (Expr e) { this.e = e; }
  public Object eval () {
    return new Boolean ((Integer) (e.eval ()) == 0);
  }
}

public class If extends Expr {
  private Expr b;
  private Expr t, e;
  public If (Expr b, Expr t, Expr e) {
    this.b = b; this.t = t; this.e = e;
  }
  public Object eval () {
    if (((Boolean) b.eval ()).booleanValue ())
      return t.eval ();
    else
      return e.eval ();
  }
}
```

3.2. Expression datatype in Haskell

$$
\begin{array}{l}
\mathbf{data}\ \mathit{Expr} :: * \ \mathbf{where} \\
\quad \begin{array}{lll}
N & :: \mathit{Int} \to & \mathit{Expr} \\
B & :: \mathit{Bool} \to & \mathit{Expr} \\
\mathit{Add} & :: \mathit{Expr} \to \mathit{Expr} \to & \mathit{Expr} \\
\mathit{IsZ} & :: \mathit{Expr} \to & \mathit{Expr} \\
\mathit{If} & :: \mathit{Expr} \to \mathit{Expr} \to \mathit{Expr} \to & \mathit{Expr}
\end{array} \\
\mathbf{data}\ \mathit{Result} = \mathit{NR}\ \mathit{Int} \mid \mathit{BR}\ \mathit{Bool} \\
\mathit{eval} :: \mathit{Expr} \to \mathit{Result} \\
\begin{array}{ll}
\mathit{eval}\ (N\ n) & = \mathit{NR}\ n \\
\mathit{eval}\ (B\ b) & = \mathit{BR}\ b \\
\mathit{eval}\ (\mathit{Add}\ x\ y) & = \mathbf{case}\ (\mathit{eval}\ x, \mathit{eval}\ y)\ \mathbf{of}\ (\mathit{NR}\ m, \mathit{NR}\ n) \to \mathit{NR}\ (m + n) \\
\mathit{eval}\ (\mathit{IsZ}\ x) & = \mathbf{case}\ (\mathit{eval}\ x)\ \mathbf{of}\ \mathit{NR}\ n \to \mathit{NB}\ (0 \equiv n) \\
\mathit{eval}\ (\mathit{If}\ x\ y\ z) & = \mathbf{case}\ (\mathit{eval}\ x)\ \mathbf{of}\ \mathit{NB}\ b \to \mathbf{if}\ b\ \mathbf{then}\ \mathit{eval}\ y\ \mathbf{else}\ \mathit{eval}\ z
\end{array}
\end{array}
$$

4. Indexing

Loss of precision again: expressions of different ‘types’.

Note the explicit tagging and untagging in Haskell, and the casts in Java. (And the lack of error-checking for ill-formed expressions!)

Can we capture the precise constraints in code, and exploit them?

4.1. Indexed datatypes

Parametrise the datatype (where parameter expresses represented type):

$$
\begin{array}{l}
\mathbf{data}\ \mathit{Expr} :: * \to * \ \mathbf{where} \\
\quad \begin{array}{lll}
N & :: \mathit{Int} \to & \mathit{Expr}\ \mathit{Int} \\
B & :: \mathit{Bool} \to & \mathit{Expr}\ \mathit{Bool} \\
\mathit{Add} & :: \mathit{Expr}\ \mathit{Int} \to \mathit{Expr}\ \mathit{Int} \to & \mathit{Expr}\ \mathit{Int} \\
\mathit{IsZ} & :: \mathit{Expr}\ \mathit{Int} \to & \mathit{Expr}\ \mathit{Bool} \\
\mathit{If} & :: \mathit{Expr}\ \mathit{Bool} \to \mathit{Expr}\ a \to \mathit{Expr}\ a \to & \mathit{Expr}\ a
\end{array}
\end{array}
$$

Note that the parameter denotes a *phantom type*:
a value of type $\mathit{Expr}\ a$ need not contain elements of type a.

4.2. Indexed programming

Specialised return types of constructors induce type constraints, which are exploited in type-checking definitions.

$$
\begin{aligned}
&\mathit{eval} :: \mathit{Expr}\ a \rightarrow a\\
&\mathit{eval}\ (N\ n) &&= n\\
&\mathit{eval}\ (B\ b) &&= b\\
&\mathit{eval}\ (\mathit{Add}\ x\ y) &&= \mathit{eval}\ x + \mathit{eval}\ y\\
&\mathit{eval}\ (\mathit{IsZ}\ x) &&= 0 \equiv \mathit{eval}\ x\\
&\mathit{eval}\ (\mathit{If}\ x\ y\ z) &&= \mathbf{if}\ \mathit{eval}\ x\ \mathbf{then}\ \mathit{eval}\ y\ \mathbf{else}\ \mathit{eval}\ z
\end{aligned}
$$

Note that all the tagging and untagging has gone, and with it the possibility of run-time errors.

By explicitly stating a property formerly implicit in the code, we have gained both in safety and in efficiency.

4.3. Indexed programming in Java

It can be done with Java (or C#) generics, but it's not so pretty:

```
public class If<T> extends Expr<T> {
  private Expr<Boolean> b;
  private Expr<T> t, e;
  public If (Expr<Boolean> b, Expr<T> t, Expr<T> e) {
    this.b = b; this.t = t; this.e = e;
  }
  public T eval () {
    if (b.eval ().booleanValue ()) return t.eval (); else return e.eval ();
  }
}
```

(Actually, not quite all the checking can be done at compile-time; sometimes some casts are still necessary.)

5. Other applications

Indexing by:

size: eg bounded vectors

shape: eg red-black trees

state: eg locking of resources

unit: eg physical dimensions

type: eg datatype-generic programming

proof: eg web applets

6. Application: indexing by size

Empty datatypes as indices (so $S\ (S\ Z)$ is a type).

$$
\begin{array}{l}
\textbf{data}\ Z \\
\textbf{data}\ S\ n
\end{array}
$$

Size-indexed type of vectors:

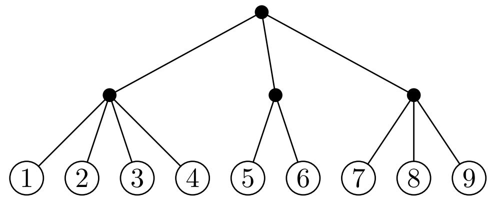
$$
\begin{array}{lll}
\textbf{data}\ \mathit{Vector} :: * \to * \to *\ \textbf{where} & & \\
\quad \mathit{VNil} & :: & \mathit{Vector}\ a\ Z \\
\quad \mathit{VCons} & :: a \to \mathit{Vector}\ a\ n \to & \mathit{Vector}\ a\ (S\ n)
\end{array}
$$

Size constraint on *vzip* is captured in the type:

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathit{vzip} :: \mathit{Vector}\ a\ n \to \mathit{Vector}\ b\ n \to \mathit{Vector}\ (a, b)\ n} \\
\mathit{vzip}\ \mathit{VNil}\ \mathit{VNil} & = \mathit{VNil} \\
\mathit{vzip}\ (\mathit{VCons}\ a\ x)\ (\mathit{VCons}\ b\ y) & = \mathit{VCons}\ (a, b)\ (\mathit{vzip}\ x\ y)
\end{array}
$$

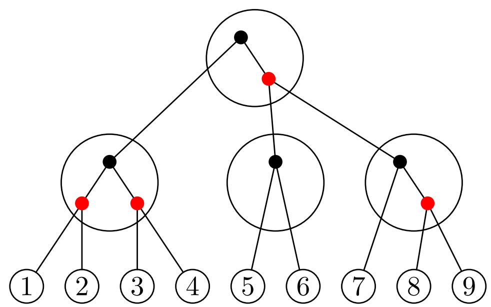
7. Application: indexing by shape

2-3-4 trees are perfectly-balanced search trees.



Representable as *red-black trees* — binary search trees in which:

- every node is coloured either red or black
- every red node has a black parent
- every path from the root to a leaf contains the same number of black nodes (enforcing approximate balance)



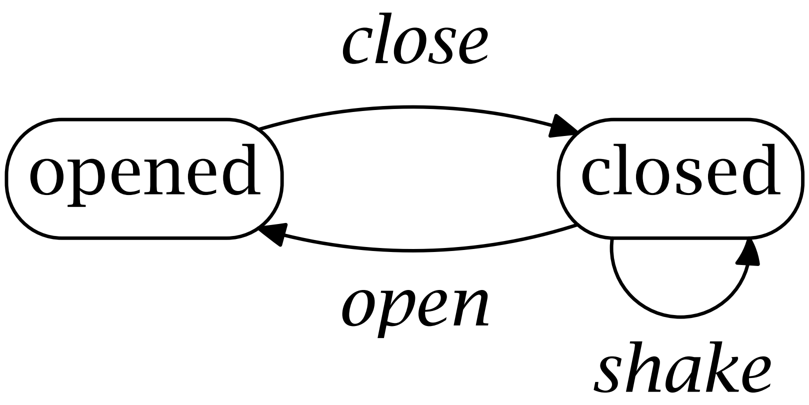
In $\mathit{RBTree}\ a\ c\ n$,

- a is the element type;
- c is the root colour;
- n is the black height.

$$
\begin{array}{l}
\textbf{data}\ R \\
\textbf{data}\ B \\
\textbf{data}\ \mathit{RBTree} :: * \to * \to * \to * \ \textbf{where} \\
\quad \begin{array}{lll}
\mathit{Empty} & :: & \mathit{RBTree}\ a\ B\ Z \\
\mathit{Red} & :: \mathit{RBTree}\ a\ B\ n \to a \to \mathit{RBTree}\ a\ B\ n & \to \mathit{RBTree}\ a\ R\ n \\
\mathit{Black} & :: \mathit{RBTree}\ a\ c\ n \to a \to \mathit{RBTree}\ a\ c'\ n & \to \mathit{RBTree}\ a\ B\ (S\ n)
\end{array}
\end{array}
$$

8. Application: indexing by state

The 'ketchup problem':



$$\begin{array}{l}
\mathbf{data}\ O \\
\mathbf{data}\ C \\
\mathbf{data}\ \mathit{Edge} :: * \to * \to *\ \mathbf{where} \\
\quad \mathit{Open} \ \ :: \mathit{Edge}\ O\ C \\
\quad \mathit{Close} \ \ :: \mathit{Edge}\ C\ O \\
\quad \mathit{Shake} :: \mathit{Edge}\ C\ C \\
\mathbf{data}\ \mathit{Path} :: * \to * \to *\ \mathbf{where} \\
\quad \mathit{Empty} :: \mathit{Path}\ s\ s \\
\quad \mathit{PCons} :: \mathit{Edge}\ x\ y \to \mathit{Path}\ y\ z \to \mathit{Path}\ x\ z \\
\mathit{scenario} :: \mathit{Path}\ O\ O \\
\mathit{scenario} = \mathit{PCons}\ \mathit{Open}\ (\mathit{PCons}\ \mathit{Shake}\ (\mathit{PCons}\ \mathit{Close}\ \mathit{Empty}))
\end{array}$$

9. Application: indexing by unit

Suppose dimensions of non-negative powers of metres and seconds:

$$\textbf{data}\ \mathit{Dim} :: * \to * \to *\ \textbf{where}$$
$$\quad D :: \mathit{Float} \to \mathit{Dim}\ m\ s$$

$$\mathit{distance} :: \mathit{Dim}\ (S\ Z)\ Z$$
$$\mathit{distance} = D\ 3.0$$

$$\mathit{time} :: \mathit{Dim}\ Z\ (S\ Z)$$
$$\mathit{time} = D\ 2.0$$

A dimensioned value is a *Float* with two type-level tags.

$$\mathit{dadd} :: \mathit{Dim}\ m\ s \to \mathit{Dim}\ m\ s \to \mathit{Dim}\ m\ s$$
$$\mathit{dadd}\ (D\ x)\ (D\ y) = D\ (x + y)$$

Now *dadd time time* is well-typed, but *dadd distance time* is ill-typed.

(More interesting to allow negative powers too, but for brevity. . .)

9.1. Type-level functions

Proofs of properties about indices:

$$
\begin{array}{l}
\mathbf{data}\ \mathit{Add} :: * \rightarrow * \rightarrow * \rightarrow *\ \mathbf{where} \\
\quad \mathit{AddZ} :: \phantom{\mathit{Add}\ m\ n\ p \rightarrow{}} \mathit{Add}\ Z\ n\ n \\
\quad \mathit{AddS} :: \mathit{Add}\ m\ n\ p \rightarrow \mathit{Add}\ (S\ m)\ n\ (S\ p)
\end{array}
$$

Used to constrain the type of dimensioned multiplication:

$$
\begin{array}{l}
\mathit{dmult} :: (\mathit{Add}\ m_1\ m_2\ m, \mathit{Add}\ s_1\ s_2\ s) \rightarrow \\
\qquad\qquad \mathit{Dim}\ m_1\ s_1 \rightarrow \mathit{Dim}\ m_2\ s_2 \rightarrow \mathit{Dim}\ m\ s \\
\mathit{dmult}\ (\_, \_)\ (D\ x)\ (D\ y) = D\ (x \times y)
\end{array}
$$

Thus, type-index of product is computed from indices of arguments.

9.2. Inferring proofs of properties

Capture the proof as a type class (multi-parameter, with functional dependency; essentially a function on types).

$$\begin{array}{l}
\textbf{class}\ \mathit{Add}\ m\ n\ p \mid m\ n \rightarrow p \\
\textbf{instance}\ \phantom{\mathit{Add}\ m\ n\ p \Rightarrow}\ \mathit{Add}\ Z\ n\ n \\
\textbf{instance}\ \mathit{Add}\ m\ n\ p \Rightarrow \mathit{Add}\ (S\ m)\ n\ (S\ p)
\end{array}$$

Now the proof can be (type-)inferred rather than passed explicitly.

$$\begin{array}{l}
\mathit{dmult} :: (\mathit{Add}\ m_1\ m_2\ m, \mathit{Add}\ s_1\ s_2\ s) \Rightarrow \\
\qquad\qquad \mathit{Dim}\ m_1\ s_1 \rightarrow \mathit{Dim}\ m_2\ s_2 \rightarrow \mathit{Dim}\ m\ s \\
\mathit{dmult}\ (D\ x)\ (D\ y) = D\ (x \times y)
\end{array}$$

Note that the type class has no methods, so corresponds to an empty dictionary; it can be optimized away.

10. Application: indexing by type

Generic programming is about writing programs parametrized by datatypes; for example, a generic data marshaller.

One implementation of generic programming manifests the parameters as some family of *type representations*.

For example, C's *sprintf* is generic over a family of *format specifiers*.

$$
\begin{array}{l}
\mathbf{data}\ \mathit{Format} :: * \to * \ \mathbf{where} \\
\quad I :: \qquad\qquad\quad \mathit{Format}\ a \to \mathit{Format}\ (\mathit{Int} \to a) \\
\quad B :: \qquad\qquad\quad \mathit{Format}\ a \to \mathit{Format}\ (\mathit{Bool} \to a) \\
\quad S :: \mathit{String} \to \mathit{Format}\ a \to \mathit{Format}\ a \\
\quad F :: \qquad\qquad\qquad\qquad\qquad\quad \mathit{Format}\ \mathit{String}
\end{array}
$$

A term of type $\mathit{Format}\ a$ is a representation of the type a,
for various types a (such as $\mathit{Int} \to \mathit{Bool} \to \mathit{String}$) appropriate for *sprintf*.

10.1. Type-indexed dispatching

The function *sprintf* *interprets* the representation, generating a function of the appropriate type:

$$
\begin{array}{l}
\mathit{sprintf} :: \mathit{Format}\ a \to a \\
\mathit{sprintf}\ \mathit{fmt} = \mathit{aux}\ \mathit{id}\ \mathit{fmt}\ \textbf{where} \\
\quad \mathit{aux} :: (\mathit{String} \to \mathit{String}) \to \mathit{Format}\ a \to a \\
\quad \mathit{aux}\ f\ (I\ \mathit{fmt}) \quad\;\; = \lambda n \to \mathit{aux}\ (f \circ (\mathit{show}\ n +\!\!+))\ \mathit{fmt} \\
\quad \mathit{aux}\ f\ (B\ \mathit{fmt}) \quad\; = \lambda b \to \mathit{aux}\ (f \circ (\mathit{show}\ b +\!\!+))\ \mathit{fmt} \\
\quad \mathit{aux}\ f\ (S\ s\ \mathit{fmt}) = \mathit{aux}\ (f \circ (s +\!\!+))\ \mathit{fmt} \\
\quad \mathit{aux}\ f\ (F) \qquad\quad\; = f\ \texttt{""}
\end{array}
$$

For example, $\mathit{sprintf}\ f\ 13\ \mathit{True} = \texttt{"Int is 13, bool is True."}$, where

$$
\begin{array}{l}
f :: \mathit{Format}\ (\mathit{Int} \to \mathit{Bool} \to \mathit{String}) \\
f = S\ \texttt{"Int is "}\ (I\ (S\ \texttt{", bool is "}\ (B\ (S\ \texttt{"."}\ F))))
\end{array}
$$

11. Application: indexing by proof

The game of *Mini-Nim*:

- a pile of matchsticks
- players take turns to remove one match or two
- player who removes the last match wins

Index positions by size and proof of destiny.

$$
\begin{array}{l}
\mathbf{data}\ \mathit{Win} \\
\mathbf{data}\ \mathit{Lose} \\[1ex]
\mathbf{data}\ \mathit{Position}\ n\ r\ \mathbf{where} \\
\quad \begin{array}{lll}
\mathit{Empty} & :: & \mathit{Position}\ Z\ \mathit{Lose} \\
\mathit{Take}1 & :: & \mathit{Position}\ n\ \mathit{Lose} \rightarrow \mathit{Position}\ (S\ n)\ \mathit{Win} \\
\mathit{Take}2 & :: & \mathit{Position}\ n\ \mathit{Lose} \rightarrow \mathit{Position}\ (S\ (S\ n))\ \mathit{Win} \\
\mathit{Fail} & :: & \mathit{Position}\ n\ \mathit{Win} \rightarrow \mathit{Position}\ (S\ n)\ \mathit{Win} \rightarrow \mathit{Position}\ (S\ (S\ n))\ \mathit{Lose}
\end{array}
\end{array}
$$

12. Adding weight

We have shown some examples in Haskell with small extensions.

This is a very lightweight approach to dependently-typed programming.

Lightweight approaches have low entry cost, but relatively high continued cost: encoding via type classes etc is a bit painful.

Tim Sheard's *Ωmega* is a cut-down version of Haskell with explicit support for GADTs:

- kind declarations
- type-level functions
- statically-generated witnesses

Xi and Pfenning's *Dependent ML* provides natural-number indices, and incorporates decision procedures for discharging proof obligations.

These are more heavyweight approaches (such as McBride et al's *Epigram*).

13. Conclusions

- generics have become mainstream
- ...but so much more than parametric polymorphism!
- *indexed programming*: a form of lightweight *dependent types*
- *Generic and Indexed Programming* project at Oxford
- perhaps algebraic datatypes will jump the gap next? (cf Scala)

14. Shameless plug...

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